EXPLICIT PRIME DENSITIES FOR THE RANK OF APPEARANCE IN LUCAS SEQUENCES

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Abstract. Let U be a Lucas sequence, p be prime, and $\rho_U(p)$ be the rank of appearance of p in U. We derive closed-form formulas for the Dirichlet density of primes p for which $d \mid \rho_U(p)$, where $d \geq 1$ is a fixed integer. Our results complete the work of Sanna (2022) by covering all U and all $d \geq 1$.

Contents

1.	Introduction	1
2.	Lemmata on cyclotomic-Kummer extensions	4
3.	The case $h = h(1)$	9
4.	Remaining cases	16
5.	Numerical data	20
References		22

1. Introduction

A *Lucas sequence* $U = U(a_1, a_2) = (U_n)_{n\geq 0}$ with non-zero parameters $a_1, a_2 \in \mathbb{Z}$ is defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = a_1U_{n+1} - a_2U_n$ for all $n \geq 0$. Given a prime number p, we denote by $\rho_U(p)$ the smallest positive n at which $p \mid U_n$. This integer is called the *rank of appearance* of p in U and exists for all $p \nmid a_2$. In this paper, we study the Dirichlet density of the set $\mathcal{R}_U(d)$ of primes $p \nmid a_2$ such that $\rho_U(p)$ is divisible by a fixed integer $d \geq 1$. This was first studied by Hasse [6, 7] for sequences $U(a + 1, a)$ with $a \in \mathbb{Z}$ square-free, $|a| \geq 2$, and d a prime number. He settled the existence and an explicit formula of the Dirichlet density. For instance, Hasse found a density of 17/24 in the case $|a| = d = 2$, and $d^2/(d^2 - 1)$ otherwise. Following his work, many authors considered this problem.

Let $f_U(X) = X^2 - a_1X + a_2$ be the characteristic polynomila of U. The problem was completely solved by Wiertelak [19, 20, 21] when $f_U(X)$ is reducible. In 1985, Lagarias [10] computed the Dirichlet density of $\mathcal{R}_U(2)$ for three Lucas sequences with irreducible polynomial. In particular, he proved that $\mathcal{R}_F(2)$ has density 2/3, where $F = U(1, -1)$ is the Fibonacci sequence. Later on, the Fibonacci case was completed for all $d \geq 1$ by

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Cubre and Rouse [3] through the study of algebraic groups. More recently, Sanna [14] made an important contribution to this problem by computing the Dirichlet density for all odd $d \geq 1$ with $3 \nmid d$ if the splitting field of f_U is $\mathbb{Q}(\sqrt{-3})$. This was done using a method of Moree [12]. Prior to Sanna's result, Ballot [1] considered the case of the splitting field being $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, and $d \in \{2, 4\}$ or $d \in \{3, 6\}$ respectively. We want to stress that a cyclotomic splitting field causes many problems when d is divisible by one of 2 or 3.

Let $a, b \in \bar{\mathbb{Q}}$ be the roots of f_U and $K = \mathbb{Q}(a)$. We denote by $\Delta = a_1^2 - 4a_2$ the discriminant of f_U and $\gamma = a/b$ its root quotient. Then, the set $\mathcal{R}_U(d)$ is equal, up to finitely many exceptions, to

$$\mathcal{R}_\gamma(d) = \{p : p \nmid 2a_2\Delta \text{ and } d \mid \operatorname{ord}_\pi(\gamma) \text{ for all } \pi \in \mathcal{O}_K \text{ and } p = \pi \cap \mathbb{Z}\},$$

where $\mathcal{O}_K$ is the ring of integers of K and $\operatorname{ord}_\pi(\gamma)$ is the order of $\gamma \bmod \pi$ in the multiplicative group $(\mathcal{O}_K/\pi)^\times$. Indeed, by [14, Lemma 4.1], we have $\rho_U(p) = \operatorname{ord}_\pi(\gamma)$. The goal of our paper is to find the Dirichlet density, if it exists, of $\mathcal{R}_U(d)$ in all the cases left by Sanna's theorem. Moreover, we want to write it in a closed-form formula, meaning that only finitely many operations are required to compute it. By the above, the sets $\mathcal{R}_U(d)$ and $\mathcal{R}_\gamma(d)$ have the same Dirichlet density, when it exists. Therefore, we study the set $\mathcal{R}_\gamma(d)$, which is more suitable for some of our calculations.

In the statement of [14, Theorem 1.1], Sanna assumed that d is odd and not divisible by 3 when $\Delta_K = -3$. However, these restrictions are not used in the proofs of the existence of the density, nor of the upper bound. Indeed, [14, Lemma 5.1] is stated without them and, while they appear in the statement of [14, Lemma 5.3], the proof does not invoke them. The proof of the main theorem only relies on these lemmas and [14, Lemmas 5.2 and 5.4]. It is in the latter that the assumption on d is required to compute a closed-form formula of the density. Let $x > 1$ and denote by $\mathcal{R}_\gamma(d; x)$ the number of primes $p \in \mathcal{R}_\gamma(d)$ with $p \leq x$. Let $K_{n,d} = K(\zeta_n, \gamma^{1/d})$, where $d \mid n$ are integers and ζ_n is a primitive n-th root of unity. We write Li, ω, and φ for the logarithmic integral, the number of distinct prime divisors, and the Euler totient functions respectively. With a slight change of the proof of [14, Lemma 5.2], the theorem can be restated, with our notation, as follows:

Theorem 1. *Let d be an integer. There exists an absolute constant $B > 0$, such that for every $x > \exp(Bd^{40})$, we have*

$$\mathcal{R}_\gamma(d; x) = \delta_\gamma(d)\operatorname{Li}(x) + \mathcal{O}_\gamma\left(\frac{d}{\varphi(d)} \cdot \frac{x(\log\log x)^{\omega(d)}}{(\log x)^{9/8}}\right),$$

where

$$\delta_\gamma(d) = \sum_{v \mid d^\infty} \sum_{u \mid d} \frac{\mu(u)(1 + [\sigma_{u,v} \text{ exists}])}{[K_{dv,uv} : \mathbb{Q}]}, \tag{1}$$

and $[\sigma_{u,v}$ *exists*$]$ *is a boolean that checks the existence of a* $\sigma_{u,v}$ *in* $Gal(K_{dv,uv}/\mathbb{Q})$ *satisfying* $\sigma_{u,v}(\sqrt{\Delta}) = -\sqrt{\Delta}$, $\sigma_{u,v}(\zeta_{dv}) = \zeta_{dv}^{-1}$, *and* $\sigma_{u,v}(\gamma^{1/uv}) = \gamma^{-1/uv}$.

In this paper, we compute closed-form formulas for the series defined in (1) in all the cases left by Sanna [14]. This includes the more difficult cases of a cyclotomic splitting field K discussed above. We note that a particular case of our results was obtained independently by Luo, Hong, and Liu [11]. They extended Sanna's result to even integers $d \geq 2$ under the assumption that K is not cyclotomic and γ is not an n-th power in K for all $n \geq 2$. They used a method similar to Cubre and Rouse [3]. To simplify our calculations, we separate $\delta_\gamma(d)$ into the smaller values

$$(2) \qquad \delta_\gamma^+(d) := \sum_{v \mid d^\infty} \sum_{u \mid d} \frac{\mu(u)}{[K_{dv,uv} : \mathbb{Q}]} \quad \text{and} \quad \delta_\gamma^-(d) := \sum_{v \mid d^\infty} \sum_{u \mid d} \frac{\mu(u)[\sigma_{u,v} \text{ exists}]}{[K_{dv,uv} : \mathbb{Q}]},$$

according to the sum $1 + [\sigma_{u,v} \text{ exists}]$. The number $\delta_\gamma^+(d)$ corresponds to the Dirichlet density of $\mathcal{R}_\gamma^+(d)$, the set of primes $p \nmid a_2$ whose rank is divisible by d and with Legendre symbol $(\Delta/p) = 1$. For $\delta_\gamma^-(d)$, we have $(\Delta/p) = -1$ and we denote by $\mathcal{R}_\gamma^-(d)$ the corresponding set.

The second section is divided in two parts. First, we give results on powers in cyclotomic extensions of a quadratic field K. For instance, we explicit a 4-th root of a norm 1 element $\gamma \in \mathbb{Q}(i)$ in Lemma 3. Next, we prove several results on *cyclotomic-Kummer extensions* of K. They are extensions of the form $K(\zeta_n, \gamma^{1/d})$, where n and $d \mid n$ are positive integers, and ζ_n is a primitive n-th root of unity. We first compute the degree over $\mathbb{Q}$ of such extensions, and then give necessary and sufficient conditions for special automorphisms to exist in their Galois groups.

In Section 3, we compute closed-form formulas for the density $\delta_\gamma(d)$ defined in (1). Let $\zeta \in K$ be a root of unity and $h(\zeta)$ be the largest n such that $\gamma \in (K^\times)^n$. We prove our formulas under the assumption that $h(1) \geq h(\zeta)$ for all roots of unity $\zeta \in K$.

In the fourth section, we show that if $h(1)$ is not maximal, then computing $\delta_\gamma(d)$ can be reduced to the computation of $\delta_{\gamma'}(d)$ for some γ' for which $h(1)$ is maximal. Therefore, we obtain that $\delta_\gamma(d)$ has a closed-form formula for all $d \geq 1$.

Finally, we provide numerical demonstration of our results in Section 5. For several choices of K, γ and d, we display the value of $\delta_\gamma(d)$ next to its experimental value computed via a SageMath [18] program.

Throughout this paper, we denote by $\gamma = a/b$ the quotient of the roots of the polynomial $f_U(X) := X^2 - a_1X + a_2$. We assume that γ is not a root of unity, as the problem becomes trivial otherwise, and that f_U is irreducible. We let $K = \mathbb{Q}(a)$, $\mu(K)$ be the set of roots of unity contained in K and $\mu_n(K)$ be the set of n-th root of unity in K. Given $\zeta \in \mu(K)$, we let $h(\zeta)$ be the largest integer $n \geq 1$ such that $\zeta\gamma \in (K^\times)^n$.

Then, a central object in our paper is the constant

$$h := \max_{\zeta \in \mu(K)} h(\zeta).$$

We use the letters p and d, n for primes numbers and positive integers respectively. Given a field L, we denote by Δ_L its absolute discriminant and by $(L^\times)^n$ the subgroup of n-th powers of L. We let d^∞ be the *supernatural number*

$$d^\infty = \prod_{p|d} p^\infty,$$

where we allow prime numbers to have infinite p-adic valuation. We use the Iverson symbol $[\mathcal{P}]$ to check the validity of a proposition $\mathcal{P}$, which is a boolean function that equals 1 if and only if $\mathcal{P}$ is true. We use ζ_n to denote a primitive n-th root of unity. We denote by (m, n) and $[m, n]$ the lcm and the gcd of integers m and n respectively. The letters φ and μ stand for the Euler totient function and the Möbius function respectively. We denote by $K_{n,d}$ the cyclotomic-Kummer extension $K(\zeta_n, \gamma^{1/d})$ of K when $d \mid n$. Finally, we write $h_m = (h, m^\infty)$ for all $m \geq 1$.

2. Lemmata on cyclotomic-Kummer extensions

This section is divided into two subsections. In the first one, we study whether γ can be expressed as a power in the cyclotomic fields $K(\zeta_n)$. In the second, we compute the degree of cyclotomic-Kummer extensions $K_{n,d} = K(\zeta_n, \gamma^{1/d})$ of K, where $d \mid n$ is a positive integer. In the case $\Delta > 0$, we give necessary and sufficient conditions for the existence of certain automorphisms in their Galois group. When $\Delta < 0$, we show that these automorphisms always exist. Below, we display two theorems found in literature that we use on many occasions.

Theorem 2. *Let K be a field and $a \in K^\times$. Then, $X^n - a$ is irreducible over K if and only if $a \notin (K^\times)^p$ for all prime $p \mid n$ and $a \notin -4(K^\times)^4$ if $4 \mid n$.*

Proof. See [9, Chapter 8, Theorem 1.6]. □

Theorem 3. *Let K be a number field and $m = \#\mu_n(K)$. Then, for all $a \in K$, the extension $K(\zeta_n, a^{1/n})/F$ is abelian if and only if $a^m \in (K^\times)^n$.*

Proof. See [9, Chapter 8, Theorem 3.2]. □

2.1. Powers in cyclotomic extensions

Throughout this subsection, we consider an arbitrary $\gamma \in K$. First, we determine at which condition a square-free γ can become a square in $K(\zeta_n)$, where $n \geq 1$. We also study the fourth roots of γ when $\Delta_K = -4$, and its cube roots when $\Delta_K = -3$.

Lemma 4. *Let $\gamma \in K \setminus (\mathbb{Q} \cup (K^\times)^2)$ and write $\gamma = u + v\sqrt{\Delta_K}$, where $u, v \in \mathbb{Q}^\times$. Let us define $N = N_{K/\mathbb{Q}}(\gamma)$ and $c := (u - \sqrt{N})/2$. Let $K_1 = \mathbb{Q}(\sqrt{c})$ and $K_2 = \mathbb{Q}(\sqrt{c/\Delta_K})$, and Δ_1 and Δ_2 their respective absolute discriminant. Then, $\sqrt{\gamma} \in K(\zeta_n)$ if and only if the following conditions hold:*

(1) $N \in (\mathbb{Q}^\times)^2$;

(2) $\Delta_1 \mid n$ *or* $\Delta_2 \mid n$.

Proof. First, assume that K is one of $\mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_3)$. In that case, the norm N can only be equal to 1. The polynomial $X^4 - 2uX^2 + 1$ annihilates $\sqrt{\gamma}$ and is irreducible over $\mathbb{Q}$ because $\gamma \notin (K^\times)^2$. It has Galois group $C_2 \times C_2$ by [15, Lemma 4.2]. The proof follows as [15, Lemma 4.6], which also deals with the non-cyclotomic cases, and we use [16, Lemma 3] for the condition on the discriminants. □

Lemma 5. *Assume that $\Delta_K = -4$. Let $\gamma \in K \setminus (\mathbb{Q} \cup (K^\times)^2)$ with $N_{K/\mathbb{Q}}(\gamma) = 1$ and write $\gamma = u + v\sqrt{-4}$, where $u, v \in \mathbb{Q}^\times$. Then, a fourth root of γ is given by*

$$\gamma^{1/4} = \left(\frac{1+\sqrt{c}}{2}\right)^{1/2} + \frac{v\sqrt{-4}}{2|v|}\left(\frac{1-\sqrt{c}}{2}\right)^{1/2}, \tag{3}$$

where $c = (u-1)/2$. Moreover, the field $L = K(\gamma^{1/4})$ is abelian over $\mathbb{Q}$ and its conductor has the form $\mathfrak{f}(L) = 2^a p_1 \cdots p_s$, where $a \in \{2, 3, 4\}$ and the p_i's are the ramified primes in L outside of 2.

Proof. Let z be the right-hand side of (3). We have

$$z^2 = \sqrt{c} + \frac{v\sqrt{-4}}{2|v|} \cdot \sqrt{1-c} = \sqrt{c} + \frac{v\sqrt{-4}}{2\sqrt{c}},$$

where we used that $c + v^2/c = 1$, because z has modulus 1. We find γ by squaring a second time and using $c - v^2/c = u$. Next, write $L = \mathbb{Q}(i, \gamma^{1/4})$. One can show that a field automorphism of $L/\mathbb{Q}$ is entirely determined by the relations

$$\sigma_{j,k} : \begin{cases} i & \mapsto \; ij, \\ \gamma^{1/4} & \mapsto \; i^k \gamma^{j/4}, \end{cases}$$

where $j \in \{-1, 1\}$ and $0 \leq k \leq 3$. Note that we used $N_{K/\mathbb{Q}}(\gamma) = 1$, i.e., $\bar{\gamma} = \gamma^{-1}$, to obtain the relations. We can see that they commute, and thus $L/\mathbb{Q}$ is abelian. Finally, we may write $L = \mathbb{Q}(i, \alpha)$, where $2\alpha^2 = 1 + \sqrt{c}$. Therefore, L is the compositum of K

and $F = \mathbb{Q}(\alpha)$. By [4, Chapter 2, Proposition 4.1.1], we have

$$\mathfrak{f}(L) = [\mathfrak{f}(K), \mathfrak{f}(F)] = [4, \mathfrak{f}(F)].$$

Next, the minimal polynomial of α is $X^4 - X^2 + (1-c)/4$. By [15, Lemma 4.2], we see that $F/\mathbb{Q}$ has Galois group C_4. Hence $\mathfrak{f}(F) = 2^b p_1 \cdots p_s$, where $b \in \{0, 2, 3\}$, by the formula proved by Spearman and Williams [17]. □

Lemma 6. *Assume that $\Delta_K = -3$. Let $\gamma \in K \setminus (\mathbb{Q} \cup (K^\times)^3)$ with $N_{K/\mathbb{Q}}(\gamma) = 1$ and write $\gamma = u + v\sqrt{-3}$, where $u, v \in \mathbb{Q}^\times$. Then, a cube root of γ is given by*

$$\gamma^{1/3} = \frac{r}{2} + \frac{v}{r^2 - 1}\sqrt{-3}, \tag{4}$$

where r is a root of $R(X) = X^3 - 3X - 2u$, which is irreducible over $\mathbb{Q}$ with three real roots and whose splitting field L is abelian. Moreover, the conductor of L is of the form

$$\mathfrak{f}(L) = 3^a p_1 \cdots p_s,$$

where $a \in \{0, 2\}$ and the p_i's are the ramified primes of L outside of 3, and $\gamma^{1/3} \in K(\zeta_n)$ if and only if $\mathfrak{f}(L) \mid n$.

Proof. The formula for $\gamma^{1/3}$ was obtained by Cavallo [2] for real quadratic fields. It turns out that the same formula holds in our case. Let z denote the right-hand side of (4). Note that $x + y\sqrt{-3}$, where $x, y \in \mathbb{Q}$, is a cube root of γ if and only if

$$x(x^2 - 9y^2) = u \quad \text{and} \quad 3y(x^2 - y^2) = v.$$

Here, $x = r/2$ and $y = v/(r^2 - 1)$. Let us first compute y^2. Since γ has norm 1, we have $u^2 + 3v^2 = 1$ and

$$y^2 = \frac{v^2}{(r^2-1)^2} = \frac{1 - u^2}{3(r^2-1)^2} = \frac{1 - \left(\frac{r^3 - 3r}{2}\right)^2}{3(r^2-1)^2},$$

where we used $2u = r^3 - 3r$. Then, expanding the numerator, we obtain

$$y^2 = \frac{-r^6 + 6r^4 - 9r^2 + 4}{12(r^2-1)^2} = \frac{-(r^2-1)^2(r^2-4)}{12(r^2-1)^2} = \frac{4 - r^2}{12}.$$

Replacing the value of y^2 in both $x(x^2 - 9y^2)$ and $3y(x^2 - y^2)$ yields u and v respectively. Hence $z^3 = \gamma$. Next, the discriminant $\mathrm{Disc}(R) = 4 \cdot 27 \cdot (1 - u^2) = (18v)^2$ of $R(X)$ is positive and a square. Thus, R has three real roots and $\mathrm{Gal}(L/\mathbb{Q}) \cong C_3$. Finally, we have $K(\gamma^{1/3}) = K(r)$ by (4). Hence $\gamma^{1/3} \in K(\zeta_n)$ if and only if $K(r) \subset K(\zeta_n)$. This is equivalent to $L = \mathbb{Q}(r)$ being a subfield of $\mathbb{Q}(\zeta_n)$ since $\mathbb{Q}(r)$ is totally real. By definition of the conductor, we obtain $\gamma^{1/3} \in K(\zeta_n)$ if and only if $\mathfrak{f}(L) \mid n$. The formula for the conductor of a cubic field was computed by Hasse [5]. (See also the work of Huard, Spearman, and Williams [8] for another characterisation.) □

2.2. Results on cyclotomic-Kummer extensions

We assume that $h = h(1)$ and write $\gamma = \gamma_0^h$ for some $\gamma_0 \in K$. We study the cyclotomic-Kummer extensions $K_{n,d} = K(\zeta_n, \gamma^{1/d})$, where $d \mid n$ is a positive integer. In the next theorem, we compute the minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$, which is essential for the degree of $K_{n,d}/\mathbb{Q}$. We write $d_0 = d/(d,h)$ and $h_0 = h/(d,h)$ throughout this subsection.

Theorem 7. *The minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$ is $X^{d_0/t} - \gamma_0^{h_0/t}$, where*

$$t := \max(m \mid \#\mu(K) : mh_m \mid d \text{ and } \gamma^{1/mh_m} \in K(\zeta_n)).$$

Proof. We use a slightly different definition of t for the proof. We observe that $mh_m \mid d$ if and only if $m \mid d_0$, and $\gamma^{1/mh_m} \in K(\zeta_n)$ if and only if $\gamma_0^{h_0/m} \in K(\zeta_n)$. In each case, the polynomial $X^{d_0/t} - \gamma_0^{h_0/t}$ annihilates $\gamma^{1/d}$ and we use Theorem 2 to prove its irreducibility. Let p be a prime divisor of d_0/t.

Assume that $v_p(t) < v_p(\#\mu(K))$. Since $p \mid d_0$, we have $ph_p \mid d$. Then, by definition of t and its maximality, we know that $\gamma_0^{h_0/t}$ is not a p-th power in $K(\zeta_n)$.

Assume that $v_p(t) = v_p(\#\mu(K))$. By contradiction, assume that $\gamma_0^{h_0/t} \in (K(\zeta_n)^\times)^p$, or equivalently that $\gamma_0^{h_0} \in K \cap (K(\zeta_n)^\times)^{p^{1+v_p(t)}}$. This is equivalent to

$$\gamma_0^{h_0 p^{v_p(t)}} = x^{p^{1+v_p(t)}},$$

for some $x \in K$, by Theorem 3, because $p^{v_p(t)}$ is the number of p-th root of unity in K. It follows that $\gamma_0^{h_0} = \zeta_{p^{v_p(t)}}^k x^p$ for some $k \in \mathbb{Z}$. However, this is a contradiction to the maximimality of h if $k \equiv 0 \pmod{p^{v_p(t)}}$, and to $h = h(1)$ otherwise.

Finally, assume that $p = 2$ and 4 divides d_0/t. In particular, we have $4 \mid n$. Thus, if we write $\gamma_0^{h_0} = -4x^4$ for some $x \in K(\zeta_n)$, then $\gamma_0^{h_0} = (2ix^2)^2$ is a square. This is a contradiction to the above. □

Corollary 8. *Let t be defined as in Theorem 7. Then,*

$$[K_{n,d} : \mathbb{Q}] = \frac{d\varphi(n)}{(d,h)t} \cdot \begin{cases} 2, & \text{if } \Delta_K \nmid n; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. We know that $K \subset \mathbb{Q}(\zeta_n)$ if and only if $\Delta_K \mid n$ by [16, Lemma 3]. Therefore, we have $[K(\zeta_n) : \mathbb{Q}] = \varphi(n) \cdot 2^{[\Delta_K \nmid n]}$ and the result follows by Theorem 7. □

In future sections, we use Lemmas 4, 5, and 6 to explicit t defined in Theorem 7. For instance, if $\Delta_K \notin \{-4, -3\}$, then $\mu(K) = \{\pm 1\}$. By Lemma 4, we have $\gamma^{1/2h_2} \in K(\zeta_n)$ if and only if $N_{K/\mathbb{Q}}(\gamma^{1/h_2}) = 1$ and one of Δ_1 or Δ_2 divides n, where the Δ_i's are defined in the lemma. Thus, $t = 2$ if $2h_2 \mid d$, $N_{K/\mathbb{Q}}(\gamma^{1/h_2}) = 1$ and one of Δ_1 or Δ_2 divides n, and $t = 1$ otherwise.

Theorem 9. *Assume $\Delta_K > 0$. Let $\sigma : K_{n,d} \to K_{n,d}$ be such that $\sigma|_{\mathbb{Q}} = \mathrm{id}$ and*

$$\sigma\colon \begin{cases} \sqrt{\Delta} & \longmapsto -\sqrt{\Delta}; \\ \zeta_n & \longmapsto \zeta_n^{-1}; \\ \gamma^{1/d} & \longmapsto \gamma^{-1/d}. \end{cases}$$

If $2h_2 \nmid d$ or $\gamma^{1/h_2} \notin (K(\zeta_n)^\times)^2$, then σ belongs to $\mathrm{Gal}(K_{n,d}/\mathbb{Q})$ if and only if the two following conditions are satisfied:

(1) $\Delta_K \nmid n$;

(2) $h_2 \nmid d$, or $h_2 \mid d$ and $N_{K/\mathbb{Q}}(\gamma^{1/h_2}) = 1$.

If $2h_2 \mid d$ and $\gamma^{1/h_2} \in (K(\zeta_n)^\times)^2$, then σ belongs to $\mathrm{Gal}(K_{n,d}/\mathbb{Q})$ if and only if the two following conditions are satisfied:

(1) $\Delta_K \nmid n$;

(2) $\sigma_0(\gamma^{1/2h_2}) = \gamma^{-1/2h_2}$,

where $\sigma_0 \in \mathrm{Gal}(K(\zeta_n)/\mathbb{Q})$ satisfies $\sigma_0(\sqrt{\Delta}) = -\sqrt{\Delta}$ and $\sigma_0(\zeta_n) = \zeta_n^{-1}$.

Proof. From the proof of [14, Lemma 4.2], we know that σ_0 exists if and only if $\Delta_L \nmid n$. Since $\sigma|_{K(\zeta_n)} = \sigma_0$, it suffices to find necessary and sufficient conditions for σ_0 to be extended into σ. By Theorem 7, let $\mu(X) = X^{d_0/t} - \gamma_0^{h_0/t}$ be the minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$. Since $K_{n,d} \cong K(\zeta_n)[X]/(\mu(X))$, we can extend σ_0 in by sending a root of μ to any root of $\sigma_0\mu$. Equivalently, we need

$$\sigma_0(\gamma_0^{h_0/t}) = \gamma_0^{-h_0/t}. \tag{5}$$

If $t = 2$, then we show that this is equivalent to $\sigma_0(\gamma^{1/2h_2}) = \gamma^{-1/2h_2}$. The direct implication is trivial by raising (5) to the (d,h')-th power, where $h' = h/h_2$. For the converse, taking the (d,h')-th root on both sides, we obtain

$$\sigma_0(\gamma^{1/2h_2(d,h')}) = \sigma_0(\gamma_0^{h_0/2}) = \zeta_{(d,h')}^k \gamma_0^{-h_0/2},$$

for some $k \in \mathbb{Z}$. Squaring both sides, we have $\sigma_K(\gamma_0^{h_0}) = \zeta_{(d,h')}^{2k}\gamma_0^{-h_0}$, which holds in K. Hence $\zeta_{(d,h')}^{2k} = 1$ and, because $2 \nmid (d,h')$, we have $\zeta_{(d,h')}^{k} = 1$ as well.

If $t = 1$, then we have two cases. If $h_2 \mid d$, then $\gamma_0^{h_0}$ is a square in K. Moreover, because $\sigma_K(\gamma) = \gamma^{-1}$, we have $\sigma_K(\gamma_0^{h_0/2}) = \pm\gamma_0^{-h_0/2}$ and it follows that $\sigma_0(\gamma_0^{h_0}) = \gamma_0^{-h_0}$ by squaring both sides. If $h_2 \nmid d$, then $\sigma_0(\gamma_0^{h_0}) = \gamma_0^{-h_0}$ if and only if $\sigma_K(\gamma^{1/h_2}) = \gamma^{-1/h_2}$. This is equivalent to the norm of γ^{1/h_2} being 1. □

We end this section with a short lemma on the existence of the automorphism $\sigma_{u,v}$ defined in Theorem 1. This will ease calculations when $\Delta < 0$, particularly when K is a field. Indeed, a direct consequence is that $\delta_\gamma(d) = 2\delta_\gamma^+(d)$.

Lemma 10. *Assume that $\Delta < 0$. Then, the Galois group of $K_{dv,uv}/\mathbb{Q}$ contains an automorphism $\sigma_{u,v}$ as defined in Theorem 1 for all $u \mid d$ and $v \mid d^\infty$.*

Proof. Let τ be the complex conjugation. Since $K_{dv,uv}/\mathbb{Q}$ is a finite Galois extension, the restriction $\tau|_{K_{dv,uv}}$ is a field automorphism of $K_{dv,uv}$. Since $\gamma \in K$ and $\Delta < 0$, we have $\tau(\gamma) = \gamma^{-1}$. It follows that $\tau(\gamma^{1/uv}) = \gamma^{-1/uv}$ and $\sigma_{u,v} = \tau|_{K_{dv,uv}}$. □

3. The case $h = h(1)$

In this section, we derive closed-form formulas for $\delta_\gamma(d)$ in the case $h = h(1)$. To do so, we compute $\delta_\gamma^+(d)$ and $\delta_\gamma^-(d)$ separatly. Recall that $\gamma = \gamma_0^h$ for some $\gamma_0 \in K$ and that $h_m = (h, m^\infty)$ for all $m \geq 2$. We define the boolean

$$\mathcal{Q} = [N_{K/\mathbb{Q}}(\gamma^{1/h_2}) = 1].$$

We first study the case $\Delta_K \notin \{-4, -3\}$, which has two subcases: when $\mathcal{Q} = 0$, which forces $\Delta_K > 0$, and when $\mathcal{Q} = 1$. Then, we consider the case $\Delta_K = -4$, and the case $\Delta_K = -3$. We start with the following lemma that generalises a formula given in the proof of [14, Lemma 5.4]:

Lemma 11. *Let $d, h \geq 1$, $e \mid d^\infty$ and $\nu \mid d^\infty$ be positive integers and define*

$$S_{d,e,h}(\nu) := \sum_{\substack{v \mid d^\infty \\ e \mid v}} \sum_{u \mid d} \frac{\mu(u)(uv,h)[\nu \mid uv]}{\varphi(dv)uv}. \tag{6}$$

If $d := D(d, \nu^\infty)$ and $(h, \nu^\infty) \mid \nu$, then

$$S_{d,e,h}(\nu) = \frac{(h, d^\infty)\nu}{d\varphi(\nu)[e, \nu(h, D^\infty)]^2} \cdot \prod_{p \mid \nu} \left(1 - \frac{(pe,\nu)^2}{p(e,\nu)^2}\right) \cdot \prod_{p \mid d} \left(\frac{p^2}{p^2 - 1}\right).$$

Proof. For $\nu = 1$, see the proof of [14, Lemma 5.4]. Let us write

$$u = u_1 u_2, \quad v = v_1 v_2 \quad \text{and} \quad e = e_1 e_2,$$

where $u_2 = (u, \nu^\infty)$, $v_2 = (v, \nu^\infty)$, and $e_2 = (e, \nu^\infty)$. We have $[\nu \mid uv] = [\nu \mid u_2 v_2]$. Using $\varphi(dv) = \varphi(d)v$ and the multiplicativity of the Möbius function μ, the Euler totient φ and the gcd, we obtain

$$S_{d,e,h}(\nu) = \sum_{\substack{v_1 \mid D^\infty \\ e_1 \mid v_1}} \sum_{\substack{v_2 \mid \nu^\infty \\ e_2 \mid v_2}} \sum_{\substack{u_1 \mid D \\ u_2 \mid \nu}} \left(\frac{\mu(u_1)(u_1 v_1, h)}{\varphi(D) u_1 v_1^2} \cdot \frac{\mu(u_2)(u_2 v_2, h)[\nu \mid u_2 v_2]}{\varphi((d,\nu^\infty)) u_2 v_2^2}\right).$$

This yields $S_{d,e,h}(\nu) = S_{D,e_1,h}(1) \cdot S_{(d,\nu^\infty),e_2,h}(\nu)$. It remains to compute $S_{(d,\nu^\infty),e_2,h}(\nu)$. Since $(h, \nu^\infty) \mid \nu$, we have

$$S_{(d,\nu^\infty),e_2,h}(\nu) = \frac{(h,\nu^\infty)}{\varphi((d,\nu^\infty))} \sum_{\substack{v_2 | \nu^\infty \\ e_2 | v_2}} \sum_{u_2 | \nu} \frac{\mu(u_2)[\nu \mid u_2 v_2]}{u_2 v_2^2},$$

as $\nu \mid u_2 v_2$ implies that $(u_2 v_2, h) = (h, \nu^\infty)$. We may now interchange the sum and the series to obtain

$$S_{(d,\nu^\infty),e_2,h}(\nu) = \frac{(h,\nu^\infty)}{\varphi((d,\nu^\infty))} \sum_{u_2 | \nu} \frac{\mu(u_2)}{u_2} \sum_{\substack{v_2 | \nu^\infty \\ \ell | v_2}} \frac{1}{v_2^2},$$

where $\ell = [e_2, \nu/u_2]$. The inner sum is computed using the Euler product formula:

$$\sum_{\substack{v_2 | \nu^\infty \\ \ell^2 | v_2}} \frac{1}{v_2^2} = \frac{1}{\ell^2} \sum_{v_2 | \nu^\infty} \frac{1}{v_2^2} = \frac{1}{\ell^2} \prod_{p | \nu} \sum_{r \geq 0} p^{-2r} = \frac{1}{\ell^2} \prod_{p|\nu} \left(\frac{p^2}{p^2 - 1} \right).$$

We obtain

$$S_{(d,\nu^\infty),e_2,h}(\nu) = \frac{(h,\nu^\infty)}{\varphi((d,\nu^\infty))} \prod_{p|\nu} \left(\frac{p^2}{p^2-1} \right) \cdot \sum_{u_2 | \nu} \frac{\mu(u_2)}{u_2 [e_2, \nu/u_2]^2}.$$

To compute the remaining sum, call it S, we expand $[e_2, \nu/u_2]$ and use properties of the gcd. We find that

$$S = \sum_{u_2|\nu} \frac{\mu(u_2)(u_2 e_2, \nu)^2}{u_2 (e_2 \nu)^2} = \frac{(e_2,\nu)^2}{(e_2\nu)^2} \sum_{u_2 | \nu} \frac{\mu(u_2)(u_2 e_2, \nu)^2}{u_2 (e_2, \nu)^2}.$$

The general term of the sum is a multiplicative function in u_2, so we apply the Euler product formula again and find

$$S = \frac{(e_2,\nu)^2}{(e_2\nu)^2} \prod_{p|\nu} \left(1 - \frac{(pe_2, \nu)^2}{p(e_2,\nu)^2} \right) = \frac{(e,\nu)^2}{(e_2\nu)^2} \prod_{p|\nu} \left(1 - \frac{(pe,\nu)^2}{p(e,\nu)^2} \right),$$

where we used that $e_2 = (e, \nu^\infty)$ on the second equality. Next, applying [14, Lemma 5.4] to $S_{D,e_1,h}(1)$, we find that

$$S_{d,e,h}(\nu) = A \cdot \prod_{p|d} \left(\frac{p^2}{p^2-1} \right) \cdot \prod_{p|\nu} \left(1 - \frac{(pe,\nu)^2}{p(e,\nu)^2} \right),$$

where A is defined by

$$A := \frac{(h, D^\infty)}{D[(h,D^\infty), e_1]^2} \cdot \frac{(h,\nu^\infty)(e,\nu)^2}{\varphi((d,\nu^\infty))(e_2\nu)^2} = \frac{(h,d^\infty)\nu}{d\varphi(\nu)} \cdot \frac{(e,\nu)^2}{(e_2\nu)^2[(h,D^\infty),e_1]^2}.$$

We used the following identities in the second equality:

$$\frac{(d,\nu^\infty)}{\varphi((d,\nu^\infty))} = \frac{\nu}{\varphi(\nu)} \quad \text{and} \quad (h, D^\infty)(h,\nu^\infty) = (h, d^\infty).$$

Finally, we can expand the lcm to find

$$\frac{(e,\nu)^2}{(e_2\nu)^2[(h,D^\infty),e_1]^2}=\frac{(e,\nu)^2(h,e_1,D^\infty)^2}{(e_2\nu)^2(h,D^\infty)^2e_1^2}=\frac{(e,\nu)^2(h,e,D^\infty)^2}{\nu^2(h,D^\infty)^2e^2}.$$

Since ν and (h,D^∞) are coprime, we have $(e,\nu)(h,e,D^\infty)=(e,\nu(h,D^\infty))$. We see that the formula becomes

$$\frac{(e,\nu(h,D^\infty))^2}{\nu^2(h,D^\infty)^2e^2}=\frac{1}{[e,\nu(h,D^\infty)]^2},$$

and the result follows. □

Remark 12. *Note that $S_{d,eh}(\nu)$, as defined in* (6), *is zero if $e\nmid d^\infty$ or $\nu\nmid d^\infty$.*

In the rest of the paper, our goal is to write $\delta_\gamma^+(d)$ and $\delta_\gamma^-(d)$ as linear combinations of the sums $S_{d,e,h}(\nu)$. Then, the density values will be in closed-forms by Lemma 11 and Remark 12. We write $S_{d,e,h}$ as a shorthand for $S_{d,e,h}(1)$.

3.1. The case $\Delta_K\notin\{-4,-3\}$ and $\mathcal{Q}=0$

Since $N_{L/\mathbb{Q}}(\gamma)=-1$, we have $h_2\geq 2$. Moreover, if $\gamma^{1/h_2}=u+v\sqrt{\Delta_K}$ with $u,v\in\mathbb{Q}^\times$, then $\mathcal{Q}=0$ is equivalent $u^2-v^2\Delta_K=-1$. Therefore, the assumption $\mathcal{Q}=0$ ensures that $2\mid h_2$ and $\Delta_K>0$. These facts are used to find a closed-form of $\delta_\gamma^-(d)$.

Theorem 13. *Assume $\mathcal{Q}=0$ and $2\mid d$. Let $e=\Delta_K/(d,\Delta_K)$. Then,*

$$\delta_\gamma^+(d)=\frac{1}{2d}\left(\frac{1}{(h,d^\infty)}+[e\mid d^\infty]\cdot\frac{(h,d^\infty)}{[(h,d^\infty),e]^2}\right)\prod_{p\mid d}\left(\frac{p^2}{p^2-1}\right).$$

Proof. Since $\mathcal{Q}=0$, and thus $N_{K/\mathbb{Q}}(\gamma^{1/h_2})=-1$, we see that $\gamma^{1/h_2}\notin(K(\zeta_n)^\times)^2$ by Lemma 4. By Corollary 8, we have

$$[K_{dv,uv}:\mathbb{Q}]=\frac{\varphi(dv)uv}{(uv,h)}\cdot 2^{[\Delta_K\nmid dv]}. \tag{7}$$

Hence, because $\Delta_K\mid dv$ if and only if $e\mid v$, we obtain

$$\delta_\gamma^+(d)=\sum_{v\mid d^\infty}\sum_{u\mid d}\frac{\mu(u)(uv,h)}{\varphi(dv)uv}\cdot\frac{1}{2^{[e\nmid v]}}=\frac{S_{d,1,h}+S_{d,e,h}}{2}.$$

Note that we used the identity $2^{1-[e\nmid v]}=1+[e\mid v]$ in the last equality. The result follows by Lemma 11 and Remark 12. □

Theorem 14. *Assume $\mathcal{Q}=0$ and $2\mid d$. Let $e=\Delta_K/(d,\Delta_K)$. Then,*

$$\delta_\gamma^-(d)=\frac{3}{2d}\left(\frac{1}{(h,d^\infty)}-[e\mid d^\infty \text{ and } h_2\nmid e]\cdot\frac{(h,d^\infty)}{[(h,d^\infty),e]^2}\right)\prod_{p\mid d}\left(\frac{p^2}{p^2-1}\right).$$

Proof. Recall that $\mathcal{Q} = 0$ implies $2 \mid h$ and $\Delta_K > 0$. By Theorem 9, we know $\sigma_{u,v}$ exists if and only if $h_2 \nmid uv$ and $\Delta_K \nmid dv$. Therefore, using Corollary 8, we obtain

$$\delta_\gamma^-(d) = \sum_{v \mid d^\infty} \sum_{u \mid d} \frac{\mu(u)(uv,h)}{\varphi(dv)uv} \cdot \frac{[e \nmid v][h_2 \nmid uv]}{2^{[e \nmid v]}}.$$

We linearise $\delta_\gamma^-(d)$ using

$$\frac{[e \nmid v][h_2 \nmid uv]}{2^{[e \nmid v]}} = \frac{1 - [h_2 \mid uv] - [e \mid v] + [e \mid v][h_2 \mid uv]}{2},$$

so that, with the notation of Lemma 11, we obtain

$$\delta_\gamma^-(d) = \frac{1}{2}\Big(S_{d,1,h} - S_{d,1,h}(h_2) - S_{d,e,h} + S_{d,e,h}(h_2) \Big).$$

By Lemma 11 with $\nu = h_2$, we see that $S_{d,1,h}(\nu) = -2S_{d,1,h}$. Similarly, when $e \mid d^\infty$, we have $S_{d,e,h}(\nu) = \big(1 - 3 \cdot [h_2 \nmid e]\big) \cdot S_{d,e,h}$. Hence

$$\delta_\gamma^-(d) = \frac{3}{2}\Big(S_{d,1,h} - [h_2 \nmid e] \cdot S_{d,e,h} \Big),$$

and the result follows by Lemma 11 and Remark 12. ☐

3.2. The case $\Delta_K \notin \{-4, -3\}$ and $\mathcal{Q} = 1$

In this case, we have to check whether γ^{1/h_2} is a square or not in $K(\zeta_n)$ in order to compute t, defined in Theorem 7. We write $\gamma^{1/h_2} = u + v\sqrt{\Delta_K}$ for some $u, v \in \mathbb{Q}^\times$, and define $c = (u-1)/2$, and Δ_1 and Δ_2 the absolute discriminants of $K_1 = \mathbb{Q}(\sqrt{c})$ and $K_2 = \mathbb{Q}(\sqrt{c/\Delta_K})$ respectively. By Lemma 4, we know that $\gamma^{1/h_2} \in (K(\zeta_n)^\times)^2$ if and only if $\Delta_1 \mid n$ or $\Delta_2 \mid n$. Moreover, we have

$$\gamma^{1/2h_2} = \sqrt{c} + \frac{v}{2}\sqrt{\frac{\Delta_K}{c}}. \tag{8}$$

Note that $c = 0$ only if $\gamma = 1$, which contradicts that γ is not a root of unity. In addition, we have $[\Delta_K, \Delta_1] = [\Delta_K, \Delta_2] = [\Delta_1, \Delta_2]$ because K, K_1 and K_2 are pairwise linearly disjoints over $\mathbb{Q}$ with $KK_1 = KK_2 = K_1K_2$.

Theorem 15. *Assume $\mathcal{Q} = 1$ and $2 \mid d$. Define*

$$e = \frac{|\Delta_K|}{(d, |\Delta_K|)} \quad \textit{and} \quad e_i = \frac{|\Delta_i|}{(d, |\Delta_i|)},$$

for all $1 \le i \le 2$. Then,

$$\delta_\gamma^+(d) = \frac{1}{2}\Big(S_{d,1,h} + S_{d,e,h} + S_{d,e_1,h}(2h_2) + S_{d,e_2,h}(2h_2) \Big).$$

In particular, if $\Delta_K < 0$, then $\delta_\gamma(d) = 2\delta_\gamma^+(d)$.

Proof. By Corollary 8, we have

$$\delta_\gamma^+(d) = \sum_{v\mid d^\infty}\sum_{u\mid d} \frac{\mu(u)(uv,h)}{\varphi(dv)uv}\cdot\frac{t}{2^{[\Delta_K\nmid dv]}}.$$

Then, we linearise the factor $t\cdot 2^{-[\Delta_K|dv]}$ using the identity $2^{1-[\Delta_K\nmid dv]} = 1+[\Delta_K\mid dv]$ and $t = 1+[2h_2\mid uv]\cdot[\Delta_1\mid dv \text{ or } \Delta_2\mid dv]$, which is obtained from Lemma 4. Next, we use the inclusion-exclusion principle on t, the equivalence $\Delta_K\mid dv$ if and only if $e\mid v$, which is also valid for (Δ_1, e_1) and (Δ_2, e_2), and the equalities $[e,e_1]=[e,e_2]=[e_2,e_3]$ that were discussed at the beginning of Subsection 3.2. We obtain

$$\frac{t}{2^{[\Delta_K\nmid dv]}} = \frac{1+[e\mid v]}{2}+\frac{[2h_2\mid uv]\cdot[e_1\mid v]}{2}+\frac{[2h_2\mid uv]\cdot[e_2\mid v]}{2}.$$

We may now linearise $\delta_\gamma^+(d)$ using the above to find the result. Finally, if $\Delta_K<0$, then $\delta_\gamma^+(d)=\delta_\gamma^-(d)$ by Lemma 10. □

We now turn our attention to $\delta_\gamma^-(d)$ in the case $\Delta_K>0$. Let us first revise the existence conditions for the automorphism $\sigma\in\mathrm{Gal}(K_{n,d}/\mathbb{Q})$ defined in Theorem 9. Since $\mathcal{Q}=1$, we see that σ exists if and only if

(1) $\Delta_K\nmid n$;

(2) $\mathcal{P}_1(n,d)=1$ or $\mathcal{P}_2(n,d)=1$,

with boolean functions

$$\mathcal{P}_1(n,d) = [2h_2\nmid d \text{ or } \forall i\in\{1,2\},\ \Delta_i\nmid n], \tag{9}$$

and

$$\mathcal{P}_2(n,d) = [2h_2\mid d]\cdot[\Delta_1\mid n \text{ or } \Delta_2\mid n]\cdot[\sigma_0(\gamma^{1/2h_2})=\gamma^{-1/2h_2}], \tag{10}$$

where $\sigma_0\in Gal(K(\zeta_n)/K)$ is such that $\sigma_0(\sqrt{\Delta})=-\sqrt{\Delta}$ and $\sigma_0(\zeta_n)=\zeta_n^{-1}$. The next lemma provides a way to compute $\sigma_0(\gamma^{1/2h_2})$ without having to consider $K(\zeta_n)$.

Lemma 16. *Assume $\mathcal{Q}=1$, $\Delta_K>0$ and $\Delta_K\nmid n$. Then, we have*

$$\mathcal{P}_2(n,d) = [2h_2\mid d]\cdot\begin{cases}[\Delta_1\mid n], & \text{if } c<0;\\ [\Delta_2\mid n], & \text{if } c>0,\end{cases}$$

where c is defined in (8).

Proof. By the proof of [14, Lemma 4.2], we know $\sigma_0\in\mathrm{Gal}(K(\zeta_n)/K)$ because $\Delta_K\nmid n$. Moreover, we only work on the boolean $[\sigma_0(\gamma^{1/2h_2})=\gamma^{-1/2h_2}]$ that appears in $\mathcal{P}_2(n,d)$. Thus, we may assume that $2h_2\mid d$ and one of Δ_1 or Δ_2 divides n. The latter ensures the existence of a square root of γ^{1/h_2} in $K(\zeta_n)$. Note that $[\Delta_1,\Delta_2]\nmid n$, otherwise Δ_K

would divide n. We saw a formula for $\gamma^{1/2h_2}$ in (8), from which we find

$$\gamma^{-1/2h_1} = -\sqrt{c} + \frac{v}{2}\sqrt{\frac{\Delta_K}{c}}.$$

By comparing them, we see that it suffices to find conditions to have $\sigma_0(\sqrt{c}) = -\sqrt{c}$ and $\sigma_0(\sqrt{c/\Delta_K}) = \sqrt{c/\Delta_K}$.

Assume that $\Delta_1 \mid n$. By definition, we have $\sqrt{c} \in \mathbb{Q}(\zeta_n)$. Since $\sigma_0|_{\mathbb{Q}(\zeta_n)}$ is the complex conjugation, for $\sqrt{c}$ to be sent to $-\sqrt{c}$, we need $c < 0$. We obtain

$$\sigma_0\left(\sqrt{\frac{c}{\Delta_K}}\right) = \sqrt{\frac{c}{\Delta_K}},$$

since $\sigma_0(\sqrt{\Delta_K}) = -\sqrt{\Delta_K}$. Hence $\sigma_0(\gamma^{1/2h_2}) = \gamma^{-1/2h_2}$. The case $\Delta_2 \mid n$ proceeds similarly, with c/Δ_K instead of c. □

Theorem 17. *Assume $\mathcal{Q} = 1$, $\Delta_K > 0$, and $2 \mid d$. Define*

$$e = \frac{\Delta_K}{(d, \Delta_K)} \quad \text{and} \quad e_i = \frac{|\Delta_i|}{(d, |\Delta_i|)},$$

for all $1 \leq i \leq 2$. Then,

$$\delta_\gamma^-(d) = \frac{S_{d,1,h} - S_{d,e,h}}{2} + (-1)^{[c>0]} \cdot \frac{S_{d,e_1,h}(2h_2) - S_{d,e_2,h}(2h_2)}{2},$$

where c is defined in (8).

Proof. Let $\nu = 2h_2$. By (9) and (10), we see that $\big(\mathcal{P}_1(dv, uv) + \mathcal{P}_2(dv, uv)\big) \cdot [\Delta_K \nmid dv]$ is equal to 1 if and only if $\sigma_{u,v}$ exists. We may write $\delta_\gamma^-(d) = S_1(d) + S_2(d)$, where

$$S_i(d) = \sum_{v \mid d^\infty} \sum_{u \mid d} \frac{\mu(u)\mathcal{P}_i(dv, uv)}{[K_{dv,uv} : \mathbb{Q}]} \cdot [\Delta_K \nmid dv],$$

for all $i \in \{1, 2\}$. We start with $S_1(d)$. Assuming $\mathcal{P}_1(dv, uv) = 1$ and $\Delta_K \nmid dv$, and using Corollary 8, we obtain a general term for $S_1(d)$ of

$$\frac{\mu(u)(uv, h)}{2\varphi(dv)uv} \cdot \mathcal{P}_1(dv, uv) \cdot [e \nmid v].$$

We used $\Delta_K \mid dv$ if and only if $e \mid v$, which is also valid for (e_1, Δ_1) and (e_2, Δ_2). Next, we may write $\mathcal{P}_1(dv, uv) = 1 - [2h_1 \mid uv \text{ and } \exists i \in \{1, 2\},\ \Delta_i \mid dv]$ and, because of the equalities $[e, e_1] = [e, e_2] = [e_1, e_2]$, we find that

$$\mathcal{P}_1(dv, uv) \cdot [e \nmid v] = \big(1 - [2h_2 \mid uv] \cdot [e_1 \mid v] - [2h_2 \mid uv] \cdot [e_2 \mid v]\big) \cdot [e \nmid v].$$

We can now use $[e \nmid v] = 1 - [e \mid v]$ to obtain

$$S_1(d) = \frac{1}{2}\bigg(S_{d,1,h} - S_{d,e,h} - S_{d,e_1,h}(2h_2) - S_{d,e_1,h}(2h_2)\bigg) + S_{d,[e_1,e_2],h}(2h_2).$$

We now turn our attention to $S_2(d)$. By Lemma 16, we have

$$\mathcal{P}_2(n,d)\cdot[\Delta_K \nmid dv] = [2h_2 \mid uv]\cdot[\Delta_K \nmid dv]\cdot\begin{cases}[\Delta_1 \mid dv], & \text{if } c<0;\\ [\Delta_2 \mid dv], & \text{if } c>0,\end{cases}$$

If $c<0$, then

$$S_2(d) = \sum_{\substack{v\mid d^\infty\\ e_1\mid v}}\sum_{u\mid d}\frac{\mu(u)(uv,h)t}{2\varphi(dv)uv}\cdot[2h_2 \mid uv]\cdot[e\nmid v] = S_{d,e_1,h}(\nu) - S_{d,[e,e_1],h}(\nu),$$

using Corollary 8 for the field degrees and Lemma 4 to show that $t=2$, where t is defined in the corollary. Going back to $\delta_\gamma^-(d) = S_1(d)+S_2(d)$, we obtain

$$\delta_\gamma^-(d) = \frac{1}{2}\Big(S_{d,1,h} - S_{d,e,h} - S_{d,e_1,h}(\nu) + S_{d,e_2,h}(\nu)\Big),$$

where we used $[e,e_1]=[e_1,e_2]$. If $c>0$, then $S_2(d) = S_{d,e_2,h}(\nu) - S_{d,[e,e_2],h}(\nu)$, and the result follows similarly. □

3.3. The case $\Delta_K = -4$

We now deal with the case of $K=\mathbb{Q}(i)$. Similar to Subsection 3.2, we have to check whether γ^{1/h_2} is a square in cyclotomic extensions $K(\zeta_n)$. Because $N_{K/\mathbb{Q}}(\gamma^{1/h_2})=1$, it is a square if and only if $\Delta_1 \mid n$ or $\Delta_2 \mid n$, where the Δ_i's were defined in the previous subsection. However, when it is a square, we also have to check whether γ^{1/h_2} is a 4-th power in $K(\zeta_n)$. By Lemma 5, if we consider $\mathfrak{f}(L)$ the conductor of $L := \mathbb{Q}(i,\gamma^{1/4h_2})$, then $\gamma^{1/4h_2}\in K(\zeta_n)$ if and only if $\mathfrak{f}(L)\mid[4,n]$.

Theorem 18. *Assume that $\Delta_K=-4$ and $2\mid d$. Define $e=4/(d,4)$,*

$$f = \frac{\mathfrak{f}(L)}{(d,\mathfrak{f}(L))} \quad \text{and} \quad e_i = \frac{|\Delta_i|}{(d,|\Delta_i|)},$$

for all $1\le i\le 2$. Then,

$$\delta_\gamma(d) = S_{d,1,h} + S_{d,e,h} + S_{d,e_1,h}(2h_2) + S_{d,e_2,h}(2h_2) + 4S_{d,f,h}(4h_2).$$

Proof. The proof is similar to the ones of Theorems 13, 14, 15, and 17, thus we only give the main steps. First, we linearise t defined in Theorem 7 as

$$t = 1 + [2h_2\mid uv]\cdot[\exists i, \Delta_i \mid dv] + 2[4h_2\mid uv]\cdot[\mathfrak{f}(L)\mid dv]$$

for the field $K_{dv,uv}$. Note that we used $4h_2\mid uv$ and u being square-free, so that $2\mid v$. Hence $4\mid dv$ and $[4,dv]=dv$. Secondly, by Corollary 8, we have

$$\frac{1}{[K_{dv,uv}:\mathbb{Q}]} = \frac{(uv,h)t}{\varphi(dv)uv}\cdot\frac{1}{2^{[4\nmid dv]}} = \frac{(uv,h)t}{\varphi(dv)uv}\cdot\frac{1+[4\mid dv]}{2}.$$

Finally, replacing this expression in $\delta_\gamma(d)$, we can linearise t and write $\delta_\gamma(d)$ in terms of sums $S_{d,e,h}(\nu)$. Also, note that we use $\Delta_i \mid dv$ if and only if $e_i \mid v$, and the same holds for the pairs $(4,e)$ and $(\mathfrak{f}(L), f)$, and $\delta_\gamma^+(d) = \delta_\gamma^-(d)$ by Lemma 10. $\square$

3.4. The case $\Delta_K = -3$

This is the last remaining case for $h = h(1)$, when $K = \mathbb{Q}(\zeta_3)$. As before, we check whether γ^{1/h_2} has a square root in $K(\zeta_n)$ or not, using the discriminants Δ_1 and Δ_2, which we defined as in Subsection 3.2. In addition, we have to check whether γ^{1/h_3} is a cube in $K(\zeta_n)$ or not. Let us write $\gamma^{1/h_3} = u + v\sqrt{-3}$ for some $u, v \in \mathbb{Q}^\times$, and L be the splitting field of $X^3 - 3X - 2u$. If $\mathfrak{f}(L)$ is the conductor of L, then $\gamma^{1/h_3} \in (K(\zeta_n)^\times)^3$ if and only if $\mathfrak{f}(L) \mid n$ by Lemma 6.

Theorem 19. *Assume that* $\Delta_K = -3$ *and* $(d,6) > 1$*. Define*

$$f = \frac{\mathfrak{f}(L)}{(d, \mathfrak{f}(L))} \quad \textit{and} \quad \tilde{e} = \min\left(\frac{|\Delta_i|}{(d, |\Delta_i|)} : 1 \leq i \leq 2\right).$$

Then, we have

$$\delta_\gamma(d) = 2^{[3|d]}\Big(S_{d,1,h} + S_{d,\tilde{e},h}(2h_2) + 2S_{d,f,h}(3h_3) + 2S_{d,[\tilde{e},f],h}(6h_6)\Big).$$

Proof. As in the proof of Theorem 18, we may skip a few details. Let t be defined as in Theorem 7 applied to the field $K_{dv,uv}$. Then, we have

$$t = \big(1 + [2h_2 \mid uv] \cdot [\exists i, \Delta_i \mid dv]\big) \cdot \big(1 + 2 \cdot [3h_3 \mid uv] \cdot [\mathfrak{f}(L) \mid dv]\big),$$

Moreover, the fields $K_1 = \mathbb{Q}(\sqrt{c})$ and $K_2 = \mathbb{Q}(\sqrt{-3c})$ have discriminants Δ_1 and Δ_2 which are equal up to a factor 3. If $3 \nmid d$, then it means only one of Δ_1 or Δ_2 may divide dv. If $3 \mid d$, then $\Delta_1 \mid dv$ if and only if $\Delta_2 \mid dv$. In both cases, we obtain

$$[\exists i, \Delta_i \mid dv] = [\tilde{e} \mid v],$$

which simplifies the expression of t. Next, by Corollary 8, we have

$$[K_{dv,uv} : \mathbb{Q}] = \frac{\varphi(dv)uv}{(uv,h)t} \cdot 2^{[3|dv]} = \frac{\varphi(dv)uv}{(uv,h)t} \cdot 2^{[3|d]},$$

because $v \mid d^\infty$. Finally, replacing this degree in the expression of $\delta_\gamma^+(d)$, linearising t, and using Lemma 10 for the equality $\delta_\gamma^+(d) = \delta_\gamma^-(d)$, we can write $\delta_\gamma(d)$ as the linear combination of the sums $S_{d,e,h}(\nu)$ given in the statement. $\square$

4. Remaining cases

Our goal is to provide a way to write $\delta_\gamma(d)$ in closed-form when $h \neq h(1)$. We first take a quick look at the case $h = h(-1)$. Then, we study the cases $h = h(\zeta)$ for $\zeta \in \mu(K)$

different than ± 1. This may only happen when $\Delta_K = -4$ or $\Delta_K = -3$. The next formula allows us to write $\delta_\gamma(d)$ exclusively in terms of $\delta_{-\gamma}(\cdot)$, which can be written in closed-form by Section 3. This is done via a formula used by Wiertelak [21] and Moree [12] that links $\operatorname{ord}_\pi(\gamma)$ and $\operatorname{ord}_\pi(-\gamma)$. We write $\zeta\gamma = \gamma_0^h$ for some $\gamma_0 \in K$.

Theorem 20. *For every $d \geq 2$, we have*

$$\delta_\gamma(d) = \begin{cases} \delta_{-\gamma}(2d) + \delta_{-\gamma}(d/2) - \delta_{-\gamma}(d), & \text{if } 2 \parallel d; \\ \delta_{-\gamma}(d), & \text{otherwise.} \end{cases}$$

In particular, if $h = h(-1)$, then the values for $\delta_{-\gamma}$ can be computed using Theorems 13, 14, 15, 17, 18, or 19.

Proof. Let $p \in \mathcal{R}_\gamma(d)$ and π be a prime ideal of $\mathcal{O}_K$ lying above p. We have

$$\operatorname{ord}_\pi(\gamma) = \begin{cases} \operatorname{ord}_\pi(-\gamma)/2, & \text{if } 2 \nmid \operatorname{ord}_\pi(\gamma); \\ 2\operatorname{ord}_\pi(-\gamma), & \text{if } 2 \parallel \operatorname{ord}_\pi(\gamma); \\ \operatorname{ord}_\pi(-\gamma), & \text{if } 4 \mid \operatorname{ord}_\pi(\gamma). \end{cases}$$

We see that $\operatorname{ord}_\pi(\gamma) = \operatorname{ord}_\pi(-\gamma)$ if $4 \mid d$ and $d \mid \operatorname{ord}_\pi(\gamma)$. Also, if $2 \nmid d$, then $d \mid \operatorname{ord}_\pi(\gamma)$ if and only if $d \mid \operatorname{ord}_\pi(-\gamma)$. Hence $\mathcal{R}_\gamma(d) = \mathcal{R}_{-\gamma}(d)$ and the result follows if $v_2(d) \neq 1$. Assume that $2 \| d$. Then, $\mathcal{R}_\gamma(d)$ is the union of the sets

$$A_1 = \{p \text{ prime} : p \nmid a_2\Delta \text{ and } 2d \mid \operatorname{ord}_\pi(\gamma)\},$$

and $A_2 = \mathcal{R}_\gamma(d) \setminus A_1$. Any $p \in A_1$ satisfies $4 \mid \operatorname{ord}_\pi(\gamma)$, so that $\operatorname{ord}_\pi(\gamma) = \operatorname{ord}_\pi(-\gamma)$. Thus, we have $A_1 = \mathcal{R}_{-\gamma}(2d)$. For A_2, we see that any p in this set satisfies

$$d \mid \operatorname{ord}_\pi(\gamma) \quad \text{and} \quad 2d \nmid \operatorname{ord}_\pi(\gamma).$$

It follows that $\operatorname{ord}_\pi(\gamma) = 2\operatorname{ord}_\pi(-\gamma)$, and that $p \in A_2$ if and only if $d \mid 2\operatorname{ord}_\pi(-\gamma)$ and $d \nmid \operatorname{ord}_\pi(-\gamma)$. We find that $A_2 = \mathcal{R}_{-\gamma}(d/2) \setminus \mathcal{R}_{-\gamma}(d)$. The result follows by taking the natural density of these sets, which exists by Sanna's theorem [14]. □

Note that these switching formulas between γ and $-\gamma$ given in Theorem 20 remain valid for δ_γ^+ and δ_γ^-, and that the proof is the same. We also use the following:

Lemma 21. *Assume $\Delta_K \in \{-4, -3\}$. Let $p \mid \Delta_K$ be prime and $d \geq 2$, $p \nmid d$. Then,*

$$[K_{p^{k+j}dv,uv} : \mathbb{Q}] = [K_{dv,uv} : \mathbb{Q}] \cdot \begin{cases} 3^{k+j-1}, & \text{if } p = 3; \\ 2^{k+j-2}, & \text{if } p = 2 \text{ and } k + j \geq 2; \\ 1, & \text{if } p = 2 \text{ and } (k, j) = (1, 0). \end{cases}$$

for all $u \mid d$, $v \mid d^\infty$, $k \geq 1$, and $j \geq 0$.

Proof. It suffices to see that $K_{p^{k+j}dv,uv}$ is the compositum of $K_{dv,uv}$ and $K(\zeta_{p^{k+j}})$, which are linearly disjoint over K. Then, we use the formula

$$[K_{p^{k+j}dv,uv} : \mathbb{Q}] = [K_{dv,uv} : \mathbb{Q}] \cdot [K(\zeta_{p^{k+j}}) : K],$$

for composita, and compute the degree of the cyclotomic extension $K(\zeta_{p^{k+j}})/K$. □

4.1. The case $h \neq h(1)$ and $\Delta_K = -4$

We may assume that $h = h(i) \neq h(\pm 1)$. Indeed, Theorem 20 already addresses the case $h = h(-1)$. Moreover, it allows us to switch between δ_γ and $\delta_{-\gamma}$. Thus, we may assume $h = h(i)$ in the following. Note that $2 \mid h$ because $h \neq h(\pm 1)$.

Theorem 22. *Assume $\Delta_K = -4$ and $h = h(i) \neq h(\pm 1)$. Let $d, k \geq 1$ be integers, $2 \nmid d$, and Δ_1 and $\mathfrak{f}(L)$ be as defined in Lemmas 4 and 5 for γ_0^{h/h_2}. Then,*

$$\delta_\gamma(2^k d) = \delta_{i\gamma}(d) \cdot \begin{cases} 1 - \dfrac{2^k}{3 \cdot 2^{m+2} h_2}, & \text{if } k \in \{1, 2\}; \\ \dfrac{8}{3 \cdot 2^{k+m} h_2}, & \text{if } k \geq 3, \end{cases}$$

where $m = [\Delta_1 \mid 8d] + [\mathfrak{f}(L) \mid 16d]$ and $\delta_{i\gamma}(d)$ is computed in [14, Theorem 1.1].

Proof. We give the main steps of the proof. First, by Lemma 10, we can write

$$\delta_\gamma(2^k d) = \sum_{j \geq 0} \sum_{v \mid d^\infty} \sum_{0 \leq l \leq 1} \sum_{u \mid d} \frac{2\mu(2^l u)}{[K_{2^{k+j}dv, 2^{j+l}uv} : \mathbb{Q}]}. \tag{11}$$

Then, by Kummer theory, we see that $K_{2^{k+j}dv,uv}$ and $K_{2^{k+j}dv,2^{j+l}}$ are linearly disjoint over $K(\zeta_{2^{k+j}dv})$. By the formula for the degree of composita, we have

$$[K_{2^{k+j}dv,2^{j+l}uv} : \mathbb{Q}] = [K_{2^{k+j}dv,uv} : \mathbb{Q}] \cdot [K_{2^{k+j}dv,2^{j+l}} : K(\zeta_{2^{k+j}dv})].$$

The degree of $K_{2^{k+j}dv,uv}/\mathbb{Q}$ is computed in Lemma 21. To compute the second degree, we need to find the largest $n \geq 1$ such that $\gamma_0^h = x^{2^n}$ in $K(\zeta_{2^{k+j}dv})$. The degree is easily computed when $k + j \leq 3$. If $k + j \geq 4$, then γ_0^{h/h_2} is a square, respectively a 4-th power, in $K(\zeta_{2^{k+j}dv})$ if and only if $\Delta_1 \mid 2^{k+j}dv$, respectively $\mathfrak{f}(L) \mid 2^{k+j}dv$, by Lemmas 4 and 5. Because Δ_1 and $\mathfrak{f}(L)$ are square-free except possibly for a factor 2^n, where $n \leq 3$ for Δ_1 and $n \leq 4$ for $\mathfrak{f}(L)$, these conditions become $\Delta_1 \mid 8d$ and $\mathfrak{f}(L) \mid 16d$. Hence the definition of m. If $k + j \geq 4$, we have

$$[K_{2^{k+j}dv,2^{j+l}} : K(\zeta_{2^{k+j}dv})] = \begin{cases} 2^{l+2-k}, & \text{if } k \in \{1, 2\}; \\ 1, & \text{if } k \geq 3, \end{cases}$$

when $j+l \leq v_2(2^m h)$. Otherwise, when $j+l > v_2(2^m h)$, we have

$$[K_{2^{k+j}dv,2^{j+l}} : K(\zeta_{2^{k+j}dv})] = \begin{cases} [2^{l+1}, 2^{j+l-v_2(2^m h)}], & \text{if } k=1; \\ 2^{j+l-v_2(2^m h)}, & \text{if } k \geq 2. \end{cases}$$

In the latter case, we first used [13, Lemma 4] with the fields $K(\zeta_{2^{k+j}dv}, (-i)^{1/2^{j+l}})$ and $K(\zeta_{2^{k+j}dv}, \gamma_0^{h_0/2^{j+l}})$, which are linearly disjoint over $K(\zeta_{2^{k+j}dv})$. Then, we used Theorem 2 to compute the degree of $K(\zeta_{2^{k+j}dv}, \gamma_0^{h_0/2^{j+l}})$. In particular, we find that the degree $A_{j,k,l} := [K_{2^{k+j}dv,2^{j+l}} : K(\zeta_{2^{k+j}dv})]$ is independent of u and v. Using Lemma 21, we obtain

$$\delta_\gamma(2^k d) = \delta_\gamma(d) \cdot \sum_{j\geq 0} \left(\frac{1}{A_{j,k,0}} - \frac{1}{A_{j,k,1}} \right) \cdot \begin{cases} 1, & \text{if } k=1 \text{ and } j=0; \\ 2^{2-j-k}, & \text{if } k+j \geq 2. \end{cases}$$

We have now obtained $\delta_\gamma(d)$ multiplied by a geometric series, the computation of which is left to the reader.

Finally, note that $K(\zeta_{dv}, \gamma^{1/uv})$ and $K(\zeta_{dv}, \gamma_0^{h/uv})$ are equal for all $u \mid d$ and $v \mid d^\infty$. Indeed, since d is odd, we have $(-i)^{1/uv}$ equal to either i or $-i$. Hence $\gamma^{1/uv} = \pm i \gamma_0^{h/uv}$ and $\delta_\gamma(d) = \delta_{i\gamma}(d)$ for all odd $d \geq 1$, which is computed in [14, Theorem 1.1]. □

4.2. The case $h \neq h(1)$ and $\Delta_K = -3$

We compute the Dirichlet density of $\mathcal{R}_\gamma(d)$ when $h \neq h(1)$ and $\Delta_K = -3$. We may assume that $h = h(\zeta_3) \neq h(\pm 1)$. The case $h = h(-1)$ follows from Theorem 20, which in addition, allows us to switch between δ_γ and $\delta_{-\gamma}$. In particular, when $h = h(\zeta_6)$ or $h = h(\zeta_6^{-1})$, we can switch to $-\gamma$, for which $h = h(\zeta_3)$ or $h = h(\zeta_3^2)$ respectively. The proofs for $h = h(\zeta_3)$ and $h = h(\zeta_3^2)$ are identical, so we present them for $h = h(\omega)$, where ω is a primitive 3-rd root of unity. Note that $h \neq h(\pm 1)$ implies $3 \mid h$.

Theorem 23. *Assume $\Delta_K = -3$ and $h = h(\omega) \neq h(\pm 1)$. Let $d, k \geq 1$ be integers, $3 \nmid d$, and $\mathfrak{f}(L)$ be as defined in Lemma 6 for γ_0^{h/h_3}. Then,*

$$\delta_\gamma(3^k d) = \delta_{\omega\gamma}(d) \cdot \begin{cases} 1 - \dfrac{1}{4 \cdot 3^m h_3}, & \text{if } k=1; \\ \dfrac{9}{4 \cdot 3^{k+m} h_3}, & \text{if } k \geq 2, \end{cases}$$

where $m = [\mathfrak{f}(L) \mid 9d]$ and $\delta_{\omega\gamma}(d)$ is computed in [14, Theorem 1.1] *and Theorem 19.*

Proof. The proof is almost identical to the one of Theorem 22, so we may skip a few details. First, we write $\delta_\gamma(3^k d)$ in the same way as (11) with the degree

$$[K_{3^{k+j}dv,3^{j+l}uv} : \mathbb{Q}] = [K_{3^{k+j}dv,uv} : \mathbb{Q}] \cdot [K_{3^{k+j}dv,3^{j+l}} : K(\zeta_{3^{k+j}dv})].$$

The fields are linearly disjoint over $K(\zeta_{3^{k+j}dv})$ by Kummer theory. Hence the degree formula. The first degree is computed in Lemma 21. For the second, note that γ_0^{h/h_3} is a cube root in $K(\zeta_{3^{k+j}dv})$ if and only if $\mathfrak{f}(L) \mid 3^{k+j}dv$ by Lemma 6. But, because $\mathfrak{f}(L)$ is square-free besides a factor 3^n, $n \leq 2$, we have $\mathfrak{f}(L) \mid 3^{k+j}dv$ if and only if $\mathfrak{f}(L) \mid 9d$ and $j+k \geq 2$. We obtained

$$[K_{3^{k+j}dv,3^{j+l}} : K(\zeta_{3^{k+j}dv})] = \begin{cases} 3^l, & \text{if } k=1 \text{ and } j+l \leq v_3(3^m h); \\ 1, & \text{if } k \geq 2 \text{ and } j+l \leq v_3(3^m h); \\ 3^{j+l-v_3(3^m h)}, & \text{if } j+l > v_3(3^m h), \end{cases}$$

where we used that $K_{3^{k+j}dv,3^{j+l}}$ is a cyclotomic field when $j+l \leq v_3(3^m h)$, and [13, Lemma 4] and Theorem 2 when $j+l > v_3(3^m h)$. The rest of the proof follows as Theorem 22. □

5. Numerical data

In this last section, we present numerical demonstration of our results. In each table, we give γ in the form $u + v\sqrt{\Delta_K}$, where $u, v \in \mathbb{Q}^\times$. For each of them, we computed h and ζ such that $\zeta\gamma = \gamma_0^h$ with $\gamma_0 \in K$. We choose the value ζ for which $h(\zeta)$ is maximal and the density values is the easiest to compute, i.e., not too many of our results are needed. For instance, if $h = h(1) = h(-1)$, then we write $\zeta = 1$. Next, we denote by $\tilde{\gamma}$ the number $\zeta\gamma$. In Tables 1 and 2, we display the value of $\tilde{\gamma}^{1/h_2}$ from which one can compute c, K_1, K_2, and their discriminants Δ_1, Δ_2.

In Table 1, we display $\mathcal{Q} = [N_{K/\mathbb{Q}}(\tilde{\gamma}^{1/h_2})]$. In Table 2, we give the value of $\mathfrak{f}(L)$, the conductor of $\mathbb{Q}(i, \tilde{\gamma}^{1/4h_2})$. Similarly, we explicit the value of $\tilde{\gamma}^{1/h_6}$ in Table 3, and give the value of $\mathfrak{f}(L)$, the conductor of $\mathbb{Q}(\omega, \tilde{\gamma}^{1/3h_6})$. Note that we choose $\tilde{\gamma}^{1/h_6}$ instead of looking at $\tilde{\gamma}^{1/h_2}$ and $\tilde{\gamma}^{1/h_3}$ separately. This does not change the values of the discriminants Δ_1 and Δ_2, nor of the conductor $\mathfrak{f}(L)$, thanks to Bézout's identity.

Finally, for each value of d in the tables, we computed $\delta_\gamma(d)$, which we display as a rational and in numerical form in the "num." column, truncated at the 6-th decimal. In the "exp." column, we provide the experimental value of $\delta_\gamma(d)$ computed via a SageMath [18] program. We tested primes up to 10^7.

γ	h	ζ	$\tilde{\gamma}^{1/h_2}$	$\mathcal{Q}$	d	$\delta_\gamma(d)$	num.	exp.
$3+\sqrt{8}$	2	1	$\frac{2+\sqrt{8}}{2}$	0	6	17/64	0.265625	0.265670
$3+\sqrt{8}$	2	1	$\frac{2+\sqrt{8}}{2}$	0	20	25/288	0.086805	0.086782
$\frac{-27-5\sqrt{29}}{2}$	2	-1	$\frac{5+\sqrt{29}}{2}$	0	8	1/6	0.166666	0.166473
$\frac{-27-5\sqrt{29}}{2}$	2	-1	$\frac{5+\sqrt{29}}{2}$	0	10	5/36	0.138888	0.139166
$\frac{17+7\sqrt{-15}}{32}$	4	1	$\frac{1-\sqrt{-15}}{4}$	1	10	5/288	0.017361	0.017287
$\frac{17+7\sqrt{-15}}{32}$	4	1	$\frac{1-\sqrt{-15}}{4}$	1	30	5/384	0.013020	0.013017

TABLE 1. The case $\Delta_K \notin \{-4,-3\}$.

γ	h	ζ	$\tilde{\gamma}^{1/h_2}$	$\mathfrak{f}(L)$	d	$\delta_\gamma(d)$	num.	exp.
$\frac{-3+2\sqrt{-4}}{5}$	1	1	$\frac{-3+2\sqrt{-4}}{5}$	20	8	1/3	0.333333	0.333427
$\frac{-3+2\sqrt{-4}}{5}$	1	1	$\frac{-3+2\sqrt{-4}}{5}$	20	10	5/72	0.069444	0.069279
$\frac{48+7\sqrt{-4}}{50}$	2	i	$\frac{3+2\sqrt{-4}}{5}$	40	10	235/1152	0.203993	0.203844
$\frac{48+7\sqrt{-4}}{50}$	2	i	$\frac{3+2\sqrt{-4}}{5}$	40	24	1/16	0.062500	0.062553
$\frac{-240-119\sqrt{-4}}{338}$	2	i	$\frac{-24+5\sqrt{-4}}{26}$	208	26	661/8064	0.075768	0.075771
$\frac{-240-119\sqrt{-4}}{338}$	2	i	$\frac{-24+5\sqrt{-4}}{26}$	208	28	35/288	0.121527	0.121457

TABLE 2. The case $\Delta_K = -4$.

γ	h	ζ	$\tilde{\gamma}^{1/h_6}$	$\mathfrak{f}(L)$	d	$\delta_\gamma(d)$	num.	exp.
$\frac{-13+3\sqrt{-3}}{14}$	1	1	$\frac{-13+3\sqrt{-3}}{14}$	7	3	3/4	0.750000	0.750058
$\frac{-13+3\sqrt{-3}}{14}$	1	1	$\frac{-13+3\sqrt{-3}}{14}$	7	14	35/288	0.121527	0.121231
$\frac{683+37\sqrt{-3}}{686}$	3	ω^2	$\frac{1+4\sqrt{-3}}{7}$	63	9	1/12	0.083333	0.083407
$\frac{683+37\sqrt{-3}}{686}$	3	ω^2	$\frac{1+4\sqrt{-3}}{7}$	63	42	1225/10368	0.118152	0.117806
$\frac{1031-520\sqrt{-3}}{1369}$	2	-1	$\frac{13+20\sqrt{-3}}{37}$	333	6	5/8	0.625000	0.624809
$\frac{1031-520\sqrt{-3}}{1369}$	2	-1	$\frac{13+20\sqrt{-3}}{37}$	333	111	407/16416	0.024792	0.024823

TABLE 3. The case $\Delta_K = -3$.

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